METRIC SPACES: FINAL EXAM 2017

DOCENT: A. V. KISELEV

Evaluation: $\min\Big(100\%, \max\big(5\,\text{prb} \times 20\% \cdot \left[\begin{smallmatrix}1.00\\1.15^{\text{top}}\end{smallmatrix}\right],\ \sum_{i=1}^{6} \text{h/w} \times 5\% + 5\,\text{prb} \times 14\% \cdot \left[\begin{smallmatrix}1.00\\1.15^{\text{top}}\end{smallmatrix}\right]\big)\Big)$.

Problem 1. The discrete metric d_0 can take two values 0 and 1. Can a metric function $\mathrm{d}_{\mathcal{X}}$ on a set $\mathcal{X}$ attain exactly *three* distinct values?

Problem 2 (top)**.** Let $S \subseteq \mathcal{X}$ be a subset of a space $\mathcal{X}$. Prove that the boundary ∂S of S is closed in $\mathcal{X}$.

Problem 3. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $\{U_i \mid i \in \mathcal{I}\}$ be a family of connected subsets $U_i \subseteq \mathcal{X}$ such that $U_i \cap U_j \neq \varnothing$ for all indexes $i, j \in \mathcal{I}$. Prove that the union $U = \bigcup_{i\in\mathcal{I}} U_i$ is connected.

Problem 4 (top)**.** Let $\mathcal{X}$ be a non-empty compact Hausdorff space and a map $f\colon \mathcal{X} \to \mathcal{X}$ be continuous. By definition, put $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_{n+1} = f(\mathcal{X}_n)$ inductively for all $n \in \mathbb{N}$.

- Prove that $A = \bigcap_{n=1}^{+\infty} \mathcal{X}_n$ is non-empty.

Problem 5. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a non-empty complete metric space. Suppose that $f, g\colon \mathcal{X} \to \mathcal{X}$ are two Banach contractions of $\mathcal{X}$. Prove that there always exists a unique point $x_0 \in \mathcal{X}$ such that $f\big(g(x_0)\big) = x_0$.

Date: April 4, 2017.

Do not postpone your success until 28 June. GOOD LUCK!